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LETTER TO THE EDITOR

Acausality of all standard relativistic wave equations with Harish–Chandra degree four

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Abstract. The long-standing conjecture of non-existence of acceptable standard relativistic wave equations for particles of spins $s \geqslant \frac{3}{2}$ is proven, for all cases with Harish–Chandra degree four. This is an important step toward the general proof since the only remaining candidate equations have higher HC degrees and therefore are expected to exhibit still worse pathologies.

Standard relativistic wave equations are partial differential equations for single mass and spin waves, of the form

$$(-\Gamma \cdot p + m)\psi = 0. \tag{1}$$

They are covariant for a representation of the Lorentz group $S(\Lambda)$, for which there exists a Hermitianising matrix η such that $(\eta\Gamma_\mu)^\dagger = \eta\Gamma_\mu$, and they allow for a positive metric i.e. a positive charge density $\psi^\dagger\eta\Gamma_0\psi$ when ψ is a positive energy solution. Their intended use is the production of elementary particles' propagators in Quantum Mechanics; a model theory having been provided by Dirac for spin-$\frac{1}{2}$ particles.

Since Dirac first proposed his higher spins equations (Dirac 1936), much time has been devoted to the study of such equations and the difficulties of those for spins $s \geqslant \frac{3}{2}$. A survey of the subject may be found in Wightman's lecture (1978), but it is worth recalling in particular the studies of Weinberg (1943), Johnson and Sudarshan (1961) and Velo and Zwanziger (1969) who pinpointed the source of all troubles with the Fierz–Pauli, or Rarita–Schwinger, spin-$\frac{3}{2}$ equation. When a minimal electromagnetic interaction is introduced, the characteristic surfaces (which in the free-field case form the light cone) open up as the intensity of the external field increases, thus allowing the propagation of tachyons. Not only that; the differential equation of motion can subsequently lose its hyperbolicity for strong enough fields. Further studies of many existing higher spin equations have shown that none remained acceptable. This has led to the statement of a conjecture (Wightman 1978) to the effect that, for all spins $s \geqslant \frac{3}{2}$, it is not possible to have at the same time a positive metric and causality of propagation of the solutions in minimal coupling.

For the purpose of proving or disproving this conjecture, we have undertaken to examine the structure of the possible Γ-matrices for all such equations (Labonté 1982, 1983a, b, c). We do this by considering a particular construction of all pertinent

† The author is presently visiting at this address while on sabbatical leave.

matrices (i.e. Γ_μ, η and the representatives of the Lorentz generators $M_{\mu\nu}$) in a basis in which the matrix J^2 giving the spin spectrum in the particle's rest frame is diagonal. Positivity of the metric is incorporated from the start and thus the question becomes: do the solutions to the equations propagate causally or not in minimal electromagnetic coupling? The results obtained up to now for waves of 'physical' spin $\frac{3}{2}$ are presented in (i) and (ii), in which the equations are described by giving their 'auxiliary' spin content; note that no value was assumed initially for the multiplicities of the spins. The number n denotes the Harish-Chandra degree of the Γ-algebra.

(i) Auxiliary spin $\frac{1}{2}$ (Labonté 1983b)

Irreducible equations exist for each $n \geqslant 4$.
All have been constructed explicitly.
All with $n = 4$ reduce to the Rarita-Schwinger equation and are acausal.

(ii) Auxiliary spins: $\frac{1}{2}$, $\frac{3}{2}$ (Labonté 1983c)

All with $n = 4$ are acausal.

Mathews and Govindarajan (1982) have recently published a modified version of the Umegawa and Visconti result concerning the bounds on the value of n, in relation to the spin content of the wavefunction. They say, on one hand, that there is no upper bound on n: this is clearly illustrated by our results (i). On the other hand, they set the lower bound

$$n \geqslant 2s_{\max} + 1$$

with $s_{\max}$ the maximum spin contained. This being the case, our results imply the following theorem.

All spin-$\frac{3}{2}$ equations with $n = 4$ are acausal.

Indeed, equations with higher auxiliary spins than those considered above would have $n \geqslant 6$. An evident corollary to this theorem is:

All equations with $n = 4$ are acausal.

We believe that obtaining these theorems is a crucial step in proving the correctness of the conjecture. Indeed, these fermion equations with Harish-Chandra degree four constitute a borderline case, in the following sense. With $n = 3$, there can never be a modification of the propagation characteristics due to minimal coupling (Velo and Zwanziger 1972). When $n = 4$, however, equation (1) leads to a generalised Klein-Gordon equation with principal part

$$\partial^\mu \partial_\mu + (\mathrm{i}e/4m^2)\left[\sum_{\mu\nu\rho}\right]\Gamma_\lambda F^{\lambda\mu} \partial^\nu \partial^\rho,$$

with the definition

$$\left[\sum_{\mu_1\mu_2\ldots\mu_n}\right]=\sum_{\text{Perm.}}(\Gamma_{\mu_1}\Gamma_{\mu_2}-g_{\mu_1\mu_2})\Gamma_{\mu_3}\ldots\Gamma_{\mu_n}.$$

Thus, it was not clear whether the $F^{\mu\nu}$ dependent term would necessarily badly modify the propagation characteristics, which for the free field are determined by $\partial^\mu\partial_\mu$. A positive answer to this question was given above, implying that the only possible good higher spin equations for fermions or bosons have $n\geqslant 5$. For $n\geqslant 5$, however, the corresponding principal part becomes totally $F^{\mu\nu}$ dependent. It is

$$\left[\sum_{\mu_1\mu_2\ldots\mu_{n-1}}\right]\Gamma_\mu F^{\mu\mu_1}\partial^{\mu_2}\partial^{\mu_3}\ldots\partial^{\mu_{n-1}}$$

and since $\partial_\mu\partial^\mu$ appears in the lower derivatives terms, still worse properties than loss of causality are to be expected (possibly equivalent to what is known as the loss or gain of constraints). A good illustration of these still worse pathologies was given by Velo and Zwanziger in their discussion of a spin-2 equation (Velo and Zwanziger 1972). It is then expected that a general argument will dispatch all these remaining cases.

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